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Rao-Nakra model with internal damping and time delay

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Abstract. In this manuscript, by using the semigroup theory, the well-posedness and exponential stability for a Rao-Nakra sandwich beam equation with internal damping and time delay is proved. The system consists of two wave equations for the longitudinal displacements of the top and bottom layers, and one Euler-Bernoulli beam equation for the transversal displacement. To the best of our knowledge from the literature, by this time, no attention was given to the asymptotic stability for Rao-Nakra model with time delay.

1. Introduction

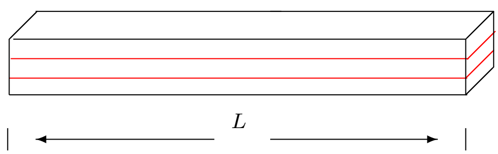
In this manuscript we consider a Rao-Nakra sandwich beam with internal damping and time delay given by

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u+v+\alpha w_x) + \mu_1 u_t + \mu_2 u_t(x,t-\tau) = 0, \tag{1}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u+v+\alpha w_x) + v_t = 0, \tag{2}$$

$$\rho h w_{tt} + EI w_{xxxx} - \alpha k(-u+v+\alpha w_x)_x + w_t = 0, \tag{3}$$

where, $(x,t,\tau) \in (0,L)\times(0,\infty)\times(0,1)$, τ is the time delay, and $\mu_1 > \mu_2 > 0$.



In this model u, v are the longitudinal displacement of the top and bottom layers and w is the transverse displacement of the beam. We consider Dirichlet-Neumann boundary conditions:

$$u(0) = u(L) = 0,$$

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$$v(0) = v(L) = 0,$$

$$w(0) = w(L) = 0,$$

$$w_x(0) = w_x(L) = 0, \tag{4}$$

and initial condition

$$(u, u_t, v, v_t, w, w_t)(x, 0) = (u_0(x), u_1(x), v_0(x), v_1(x), w_0(x), w_1(x)). \tag{5}$$

Rao-Nakra sandwich beam is derived of the following general three-layer laminated beam and plate models developed in (1999) by Liu-Trogdon-Yong [14]:

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \tau = 0, \tag{6}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \tau = 0, \tag{7}$$

$$\rho h w_{tt} + EI w_{xxxx} - G_1 h_1 (w_x + \phi_1)_x - G_3 h_3 (w_x + \phi_3)_x - h_2 \tau_x = 0, \tag{8}$$

$$\rho_1 I_1 \phi_{1,tt} - E_1 I_1 \phi_{1,xx} - \frac{h_1}{2}\tau + G_1 h_1 (w_x + \phi_1) = 0, \tag{9}$$

$$\rho_3 I_3 \phi_{3,tt} - E_3 I_3 \phi_{3,xx} - \frac{h_3}{2}\tau + G_3 h_3 (w_x + \phi_3) = 0. \tag{10}$$

The physical parameters $h_i, \rho_i, E_i, G_i, I_i > 0$ are the thickness, density, Young's modulus, shear modulus, and moments of inertia of the i-th layer for $i = 1, 2, 3$, from bottom to top, respectively. In addition, $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$ and $EI = E_1 I_1 + E_3 I_3$.

The Rao-Nakrao system [24]:

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0,$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0,$$

$$\rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0,$$

is obtained from (6)-(10) when we consider the core material to be linearly elastic, i.e., $\tau = 2G_2\gamma$ with the shear strain:

$$\gamma = \frac{1}{2h_2}(-u + v + \alpha w_x) \text{ and } \alpha = h_2 + \frac{1}{2}(h_1 + h_3),$$

where $k := \dfrac{G_2}{h_2}$, the shear modulus $G_2 = \dfrac{E_2}{2(1+\nu)}$, and $-1 < \nu < \dfrac{1}{2}$ is Poisson ratio.

When the extensional motion of the bottom and top layers is neglected, we obtain the two-layer laminated beam model proposed by Hansen and

Spies [10]:

$$\varrho w_{tt} + G(\psi - w_x)_x = 0, \text{ in } (0, L) \times (0, \infty), \tag{11}$$

$$I_\varrho(3s_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - u_x) = 0, \text{ in } (0, L) \times (0, \infty), \tag{12}$$

$$3I_\varrho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\mu s + 4\delta s_t = 0, \text{ in } (0, L) \times (0, \infty), \tag{13}$$

where $\varrho, G, I_\varrho, D, \gamma$ and δ are positive constants and represent density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. The function $w(x, t)$ denotes the transversal displacement, $\psi(x, t)$ represents the rotational displacement, and $s(x, t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. In (11)-(13), the first two equations are related to the well-known Timoshenko system, and the third one describes the dynamic of the slip. For this model, when the internal damping is added for all three equations, was proved the exponential stability in [25]. In the last years, several studies have been made in the context of stabilization of laminated beam. See for instance [18] and references therein, where authors considered a thermoelastic laminated beam with nonlinear weights and time-varying delay.

The control of Partial Differential Equations with delay has become an attractive area of research because time delays so often arise in many physical, chemical, biological and economical phenomena, see [29] and references therein. Whenever energy is physically transmitted from one place to another, there is a delay associated with the transmission, see [28]. Time delay is the property of a physical system by which the response to an applied force is delayed in its effect. The central question is that delays source can destabilize a system that is asymptotically stable in the absence of delays, see [2–4, 30] and references therein.

By Energy Method, in [17] was proved the exponential decay of solution for a wave equation with the delay term in the boundary or internal feedbacks. By semigroup approach in [26] was proved the well-posedness and exponential stability for a wave equation with frictional damping and non-local time-delayed condition. For a wave equation with non-constant delay and nonlinear weights, global existence and energy decay of solutions was considered in [1]. For wave equation with boundary time-varying delay see [16]. In [27] was given the uniform asymptotic stability of solution for linear neutral differential equations of the third order with delay.

The following Rao-Nakra model with internal damping and Kelvin-Voigt damping was considered in [12]:

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) - a_1 u_{txx} + a_2 u_t = 0, \tag{14}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) - b_1 u_{txx} + b_2 u_t = 0, \tag{15}$$

$$\rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x - c_1 w_{txxxx} + c_2 u_t = 0, \tag{16}$$

where $a_i, b_i, c_i \geq 0$, $i = 1, 2$. Authors showed that (14)-(16) is unstable if one damping is only imposed on the beam equation, beyond this, the exponential stability holds when all three displacements are damped while polynomial stability holds when just two of the three equations are damped.

For $a_1, b_1, c_1 = 0$ we recover the system (1)-(3) without time delay. In this case, in [13] was proved the polynomial stability when the damping is just on one of three wave equations. Exponential stability was obtained by Özkan Özer-Hansen [19] when standard boundary damping is imposed on one end of the beam for all three displacements. For boundary controllability problems of the Rao-Nakra beam equation (multi layers, $\alpha > 0$) we cite for instance [8, 9, 20, 23]. Exact controllability results for the multilayer Rao-Nakra plate system with locally distributed control in a neighborhood of a portion of the boundary were obtained by the method of Carlman estimates was considered in [6, 7].

Our purpose in this paper is the asymptotic behavior of the solution. The plan of the paper is as follows. First, in the section 2 is presented the well-posedness of the problem (1)-(2) by using semigroup approach. In the section 3 the exponential stability of the C_0-semigroup of contractions on a appropriated Hilbert space is proved by Gearhart's theorem [5].

2. SEMIGROUP SETTING

Proceeding as Nicaise and Pignotti [17] we introduce the following dependent variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad \rho \in (0, 1).$$

The new variable satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0$$

and then, the problem (1)-(3) can be rewritten as

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u+v+\alpha w_x) + \mu_1 u_t + \mu_2 z(x,1,t) = 0, \tag{17}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u+v+\alpha w_x) + v_t = 0, \tag{18}$$

$$\rho h w_{tt} + EI w_{xxxx} - \alpha k(-u+v+\alpha w_x)_x + w_t = 0, \tag{19}$$

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0. \tag{20}$$

The above system is subjected to initial conditions (5) and

$$\begin{aligned} &z(x,\rho,0) = f_0(x,-\tau\rho),\ f_0 \text{ belongs to a suitable Sobolev space,}\\ &z(x,0,t) = u_t(x,t),\\ &z(x,1,t) = u_t(x,t-\tau). \end{aligned}$$

In order to use the semigroup approach, we write our system (17)-(20) as a first-order system. As in [17], we denote $U = (u,u',v,v',w,w',z)^T$ a vector function, with $u' = u_t$, $v' = v_t$, $w' = w_t$, and we obtain

$$\frac{\mathrm{d}\,U}{\mathrm{d}\,t} = \mathcal{A}U, \quad U\Big|_{t=0} = (u_0,u_1,v_0,v_1,w_0,w_1,f_0)^T = U_0.$$

The operator $\mathcal{A}$ is defined as

$$\mathcal{A}U = \begin{pmatrix} u' \\ \dfrac{1}{\rho_1 h_1}\left[E_1 h_1 u_{xx} + k(-u+v+\alpha w_x) - \mu_1 u' - \mu_2 z(x,1,t)\right] \\ v' \\ \dfrac{1}{\rho_3 h_3}\left[E_3 h_3 v_{xx} - k(-u+v+\alpha w_x) - v'\right] \\ w' \\ \dfrac{1}{\rho h}\left[-EI w_{xxxx} + \alpha k(-u+v+\alpha w_x)_x - w'\right] \\ \dfrac{-1}{\tau} z_\rho \end{pmatrix},$$

Hereafter, we denote by $L^2(0,L)$ the usual Lebesgue space with the inner product and norm

$$\langle \varphi\,,\,\psi\rangle = \int_0^L \varphi\psi\,\mathrm{d}\,x, \quad ||\varphi|| = \left\{\int_0^L |\varphi|^2\,\mathrm{d}\,x\right\}^{\frac{1}{2}}.$$

For a non-negative integer m, $H^m(0,L)$ denotes the usual Sobolev space with the norma $||\cdot||_m$. We denote

$$L^2(0,1;L^2(0,L)) = \{z \in L^2(0,L) \ : \ \int_0^1 ||z||^2 \,\mathrm{d}\,\rho < \infty\}$$

and introduce the phase space $\mathcal{H}$ given by

$$\mathcal{H} = [H_0^1(0,L) \times L^2(0,L)]^3 \times L^2(0,1;L^2(0,L))$$

equipped with the inner product given by

$$\begin{aligned}
&\langle V_1, V_2\rangle_{\mathcal{H}} = \\
&= \rho_1 h_1 \langle \upsilon_2, \vartheta_2\rangle + E_1 h_1 \langle \upsilon_{1,x}, \vartheta_{1,x}\rangle + \rho_3 h_3 \langle \upsilon_4, \vartheta_4\rangle + E_3 h_3 \langle \upsilon_{3,x}, \vartheta_{3,x}\rangle) \\
&+ \rho h \langle \upsilon_6, \vartheta_6\rangle + EI \langle \upsilon_{5,xx}, \vartheta_{5,xx}\rangle + k\langle -\upsilon_1 + \upsilon_3 + \alpha \upsilon_{5,x}, -\vartheta_1 + \vartheta_3 + \alpha\vartheta_{5,x}\rangle \\
&+ \eta \int_0^1 \langle \upsilon_7, \vartheta_7\rangle \,\mathrm{d}\,\rho,
\end{aligned}$$

for

$$\begin{aligned}
V_1 &= (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6, \upsilon_7)^T, \\
V_2 &= (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6, \vartheta_7)^T \in \mathcal{H}
\end{aligned}$$

and η is a positive constant satisfying

$$\mu_2 < \frac{\eta}{\tau} < 2\mu_1 - \mu_2. \tag{21}$$

Then, $||U||_{\mathcal{H}}$ is given by

$$\begin{aligned}
||U||^2_{\mathcal{H}} = &\rho_1 h_1 ||\upsilon_2||^2 + E_1 h_1 ||\upsilon_{1,x}||^2 + \rho_3 h_3 ||\upsilon_4||^2 + E_3 h_3 ||\upsilon_{3,x}||^2 \\
&+ EI ||\upsilon_{5,xx}||^2 + \rho h ||\upsilon_6||^2 + k||-\upsilon_1 + \upsilon_3 + \alpha \upsilon_{5,x}||^2 + \eta ||\upsilon_7||^2_{L^2(0,1;L^2(0,L))}.
\end{aligned}$$

From now and on, we consider

$$\mathcal{A} \, : \, D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

whose domain is

$$D(\mathcal{A}) = \left\{ U \in \mathcal{H} \;\middle|\; \mathrm{u}_4, \mathrm{u}_5 \in H^2(0,L), \ z \in H_0^1(0,1;L^2(0,L)). \right\}$$

Lemma 1. *The operator $\mathcal{A}$ defined above is dissipative, that is, there exist positive constants a, b such that*

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\left[a||u'||^2 + ||v'||^2 + ||w'||^2 + b||z(x,1,t)||^2\right] \le 0. \tag{22}$$

Proof.

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =$$

$$= \Big\langle \begin{pmatrix} u' \\ \dfrac{1}{\rho_1 h_1}\left[E_1 h_1 u_{xx} + k(-u+v+\alpha w_x) - \mu_1 u' - \mu_2 z(x,1,t)\right] \\ v' \\ \dfrac{1}{\rho_3 h_3}\left[E_3 h_3 v_{xx} - k(-u+v+\alpha w_x) - v'\right] \\ w' \\ \dfrac{1}{\rho h}\left[-EI w_{xxxx} + \alpha k(-u+v+\alpha w_x)_x - w'\right] \\ \dfrac{-1}{\tau} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ u' \\ v \\ v' \\ w \\ w' \\ z \end{pmatrix} \Big\rangle$$

$$\begin{aligned}
&= \rho_1 h_1 \langle \frac{1}{\rho_1 h_1}\left[E_1 h_1 u_{xx} + k(-u+v+\alpha w_x) - \mu_1 u' - \mu_2 z(x,1,t)\right], u' \rangle \\
&\quad + \rho_3 h_3 \langle \frac{1}{\rho_3 h_3}\left[E_3 h_3 v_{xx} - k(-u+v+\alpha w_x) - v'\right], v' \rangle \\
&\quad + EI \langle \frac{1}{\rho h}\left[-EI w_{xxxx} + \alpha k(-u+v+\alpha w_x)_x - w'\right], w' \rangle \\
&\quad + E_1 h_1 \langle u'_x , u_x \rangle + E_3 h_3 \langle v'_x , v_x \rangle + \rho h \langle w'_{xx} , w_{xx} \rangle \\
&\quad + k\langle (-u' + v' + \alpha w_x), (-u + v + \alpha w_{,x}) \rangle + \eta \int_0^1 \langle \frac{-1}{\tau} z_\rho , z \rangle \,\mathrm{d}\,\rho.
\end{aligned}$$

Performing integration by parts we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =$$

$$\begin{aligned}
&= -E_1 h_1 \langle u_x , u' - k\langle -u+v+\alpha w_x , -u' \rangle - \mu_1 ||u'||^2 - \mu_2 \langle z(x,1,t) , u' \rangle \\
&\quad - E_3 h_3 \langle v_x , v' \rangle - k\langle -u+v+\alpha w_x , v' \rangle - ||v'||^2 \\
&\quad - EI \langle w_{xx} , w_{xx} \rangle - k\langle -u+v+\alpha w_x , \alpha\, w'_x \rangle - ||w'||^2 \\
&\quad + E_1 h_1 \langle u'_x , u_x \rangle + E_3 h_3 \langle v'_x , v_x \rangle + EI \langle w'_{xx} , w'_{xx} \rangle \\
&\quad + k\langle (-u' + v' + \alpha w_x), (-u + v + \alpha w_{,x}) \rangle + \eta \int_0^1 \langle \frac{-1}{\tau} z_\rho , z \rangle \,\mathrm{d}\,\rho.
\end{aligned}$$

Adding and taking into account

$$\begin{aligned}
\int_0^1 \langle \frac{-1}{\tau} z_\rho , z \rangle \,\mathrm{d}\,\rho &= -\frac{1}{\tau} \int_0^L \int_0^1 \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\,\rho} |z|^2 \,\mathrm{d}\,\rho \,\mathrm{d}\,x \\
&= -\frac{1}{2\tau} \int_0^L |z(x,1,t)|^2 \,dx + \frac{1}{2\tau} \int_0^L |z(x,0,t)|^2 \,\mathrm{d}\,x \\
&= -\frac{1}{2\tau} ||z(x,1,t)||^2 + \frac{1}{2\tau} ||u'||^2.
\end{aligned}$$

that is,

$$\eta \int_0^1 \langle \frac{-1}{\tau} z_\rho \, , \, z\rangle \, \mathrm{d}\, \rho = -\frac{\eta}{2\tau}||z(x,1,t)||^2 + \frac{\eta}{2\tau}||z(x,0,t)||^2$$

we obtain

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\mu_1 ||u'||^2 - \mu_2 \langle z(x,1,t)\, , \, u'\rangle \\ &\quad - ||v'||^2 - ||w'||^2 - \frac{\eta}{2\tau}||z(x,1,t)||^2 + \frac{\eta}{2\tau}||z(x,0,t)||^2.\end{aligned}$$

Applying Young's inequality and remembering $z(x,0,t) = u_t(x,t)$ a straight forward calculation leads to

$$\langle \mathcal{A}U \, , \, U\rangle_{\mathcal{H}} \leq -a||u'||^2 - ||v'||^2 - ||w'||^2 - b||z(x,1,t)||^2$$

where $a \stackrel{\text{def}}{=} \left(\mu_1 - \frac{\mu_2}{2} - \frac{\eta}{2\tau}\right) > 0$ and $b \stackrel{\text{def}}{=} \left(\frac{\eta}{2\tau} - \frac{\mu_2}{2}\right) > 0$, from (21). Then, taking the real part we concludes the proof of lemma.

□

The fundamental property of operator $\mathcal{A}$ is:

Theorem 1. *The operator $\mathcal{A}$ defined above is the infinitesimal generator of a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in the Hilbert space $\mathcal{H}$.*

Proof. It is obvious that $D(\mathcal{A})$ is dense in $\mathcal{H}$. From (22), the operator $\mathcal{A}$ is dissipative. Moreover, assume that $0 \in \sigma(\mathcal{A})$ the spectrum of the operator $\mathcal{A}$. Then there exists a sequence

$$U_n = (u_n, u_n', v_n, v_n', w_n, w_n', z_n)^T \in D(\mathcal{A})$$

with $||U_n||_{\mathcal{H}} = 1$ such that $\mathcal{A}U_n = o(1)$, i.e., $\mathcal{A}U_n \to 0$ in $\mathcal{H}$.

From

$$E_1h_1||u_{n,x}'||^2 + E_3h_3||v_{n,x}'||^2 + EI||w_{n,xx}'||^2 + \frac{\eta}{\tau^2}||z_{n,\rho}||^2_{L^2(0,1;L^2(0,L))} \leq ||\mathcal{A}U_n||^2,$$

we get

$$||u_{n,x}'||^2 = o(1), \; ||v_{n,x}'||^2 = o(1), \; ||w_{n,xx}'||^2 = o(1), \; ||z_{n,\rho}||^2_{L^2(0,1;L^2(0,L))} = o(1)$$

and applying Poincaré inequality we obtain

$$||u_n'|| = o(1), \; ||v_n'|| = o(1), \; ||w_n'|| = o(1), \; ||z_n||_{L^2(0,1;L^2(0,L))} = o(1). \tag{23}$$

By other hand, for $V_n = (0, u_n, 0, v_n, 0, w_n, 0)^T$,

$$o(1) = -\langle \mathcal{A}U_n, V_n\rangle = E_1h_1||u_{n,x}||^2 + E_3h_3||v_{n,x}||^2 + EI||w_{n,xx}||^2.$$

Then

$$||u_{n,x}|| = o(1),\ ||v_{n,x}|| = o(1),\ ||w_{n,x}|| = o(1),\ ||w_{n,xx}|| = o(1). \tag{24}$$

As

$$||U_n||^2_{\mathcal{H}} = \rho_1 h_1 ||u'_n||^2 + E_1 h_1 ||u_{n,x}||^2 + \rho_3 h_3 ||v'_n||^2 + E_3 h_3 ||v_{n,x}||^2 + EI ||w_{n,xx}||^2$$
$$+ \rho h ||w'_n||^2 + k|| - u_n + v_n + \alpha w_{n,x}||^2 + ||z_{n,\rho}||^2_{L^2(0,1;L^2(0,L))},$$

from (23) and (24) we conclude that $||U_n||_{\mathcal{H}} = o(1)$, which contradicts our assumption. Hence, $0 \in \mathbb{C} - \sigma(\mathcal{A})$ the resolvent set of $\mathcal{A}$. By Theorem 1.2.4 in [15], $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $e^{\mathcal{A}t}$ in the Hilbert space $\mathcal{H}$. □

The well-posedeness is given by the following result:

Theorem 2. *Let $U_0 \in \mathcal{H}$, then there exists a unique weak solution $U = (u, v, w)$ of problem* (1)-(3) *satisfying*

$$U \in C([0, +\infty); \mathcal{H}). \tag{25}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C([0, +\infty); D(\mathcal{A})) \cap C^1([0, +\infty); \mathcal{H}). \tag{26}$$

Proof. From semigroup theory, see [21], $U(t) = e^{t\mathcal{A}}U_0$ is the unique solution of problem (1)-(3) satisfying (25) and (26). □

3. Exponential stability

In this section, we investigate the exponential stability of the solution of system (1)-(3). The necessary and sufficient conditions for the exponential stability of the C_0-semigroup of contractions on a Hilbert space were obtained by Gearhart [5] and Huang [11] independently, see also Pruss [22]. We will use the following result due to Gearhart.

Theorem 3. *Let $\rho(A)$ be the resolvent set of the operator A and $S(t) = e^{tA}$ be the C_0-semigroup of contractions generated by A. Then, $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} = \{i\beta : \beta \in \mathbb{R}\} \subset \rho(A), \tag{27}$$

$$\limsup_{|\beta|\to\infty} \|(i\beta I - A)^{-1}\| < \infty. \tag{28}$$

The main result of this manuscript is the following theorem.

Theorem 4. *The semigroup $S(t) = e^{t\mathcal{A}}$ generated by $\mathcal{A}$ is exponentially stable.*

Proof. It is sufficient to verify (27) and (28). We prove by contradiction argument. If (27) is not true, then there exists a $\beta \in \mathbb{R}$ such that $\beta \neq 0$ and $i\beta$ is in the spectrum $\mathcal{A}$. By spectral theory, using the compact immersion of $D(\mathcal{A})$ in $\mathcal{H}$, there is a vector function

$$U = (u, u', v, v', w, w', z)^T \in D(\mathcal{A}), \text{ with } \|U\|_{\mathcal{H}} = 1$$

such that $\mathcal{A}U = i\beta U$, which is equivalent to

$$u' = i\beta u, \tag{29}$$

$$\frac{1}{\rho_1 h_1}\left[E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) - \mu_1 u' - \mu_2 z(x, 1, t)\right] = i\beta u', \tag{30}$$

$$v' = i\beta v,$$

$$\frac{1}{\rho_3 h_3}\left[E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) - v'\right] = i\beta v', \tag{31}$$

$$w' = i\beta w,$$

$$\frac{1}{\rho h}\left[-EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x - w'\right] = i\beta w', \tag{32}$$

$$\frac{-1}{\tau} z_\rho = i\beta z. \tag{33}$$

Multiplying (29) by u', integrating on $(0, L)$ and using Young's inequality we have

$$||u'||^2 = i\beta\langle u, u'\rangle \leq -\frac{1}{2}\beta^2||u||^2 + \frac{1}{2}||u'||^2,$$

from where it follows that

$$\frac{1}{2}\beta^2||u||^2 + \frac{1}{2}||u'||^2 \leq 0,$$

then, we obtain $u = u' = 0$ a.e. in $L^2(0, L)$. Similarly, we have $v = v' = w = w' = 0$ a.e. in $L^2(0, L)$.

Multiplying (33) by z and integrating in $(0, 1)$ we obtain

$$i\beta\int_0^1 ||z||^2 \,\mathrm{d}\,\rho = -\frac{1}{\tau}\int_0^1 \langle z_\rho\,,\, z\rangle \,\mathrm{d}\,\rho = -\frac{\eta}{2\tau}||z(x, 1, t)||^2 + \frac{\eta}{2\tau}||u'||^2,$$

that is,

$$i\beta \int_0^1 ||z||^2 \,\mathrm{d}\,\rho + \frac{\eta}{2\tau}||z(x,1,t)||^2 = 0.$$

Taking the imaginary part, we obtain

$$||z||^2_{L^2(0,1;L^2(0,L))} = 0.$$

From (30), (31), (32) we deduce

$$E_1 h_1 u_x = -k\alpha w, \;\; E_3 h_3 v_x = k\alpha w, \;\; EI w_{xx} = \alpha^2 k w,$$

then

$$u_x = v_x = w_{xx} = 0, \quad \text{a.e. in} L^2(0,L).$$

From $w_x \in H^1(0,L)$ and boundary condition (4) we can apply Poincaré inequality for w_x and obtain $||w_x|| \le C||w_{xx}||$ that implies $w_x = 0$ a.e. in $L^2(0,L)$. Then we have $\|U\|_{\mathcal{H}} = 0$ that contradicts with $\|U\|_{\mathcal{H}} = 1$ and consequently (27) holds.

To prove (28) we use contradiction argument again. If (28) is not true, there exists a real sequence β_n, with $\beta_n \to \infty$ and a sequence of vector functions $V_n \in \mathcal{H}$ that satisfies

$$\frac{\|(\lambda_n I - \mathcal{A})^{-1} V_n\|_{\mathcal{H}}}{\|V_n\|_{\mathcal{H}}} \ge n, \quad \text{where } \lambda_n = i\beta_n.$$

Hence

$$\|(\lambda_n I - A)^{-1} V_n\|_{\mathcal{H}} \ge n\|V_n\|_{\mathcal{H}}. \tag{34}$$

Since $\lambda_n \in \rho(\mathcal{A})$ it follows that there exists a unique sequence

$$U_n = (u_n, u_n', v_n, v_n', w_n, w_n', z_n)^T \in D(\mathcal{A})$$

with unit norm in $\mathcal{H}$ such that

$$(\lambda_n I - \mathcal{A})^{-1} V_n = U_n.$$

As $V_n = \lambda_n U_n - \mathcal{A} U_n$ we have from (34) that

$$\|V_n\|_{\mathcal{H}} \le \frac{1}{n}$$

and then $V_n \to 0$ strongly in $\mathcal{H}$ as $n \to \infty$.

Taking the inner product of V_n with U_n we have

$$\lambda_n \|U_n\|^2_{\mathcal{H}} - \langle \mathcal{A} U_n, U_n\rangle_{\mathcal{H}} = \langle V_n, U_n\rangle_{\mathcal{H}}. \tag{35}$$

Using (22) together with (35) and taking the real part, we have

$$\left[a||u_n'||^2 + ||v_n'||^2 + ||w_n'||^2 + b||z_n(x,1,t)||^2\right] = \mathrm{Re}\langle V_n, U_n\rangle_{\mathcal{H}}.$$

As U_n is bounded and $V_n \to 0$ we obtain

$$u_n' \to 0 \quad \text{as} \;\; n \to \infty, \tag{36}$$

$$v_n' \to 0 \quad \text{as} \;\; n \to \infty, \tag{37}$$

$$w_n' \to 0 \quad \text{as} \;\; n \to \infty, \tag{38}$$

$$z_n(x,1,t) \to 0 \quad \text{as} \;\; n \to \infty. \tag{39}$$

Let $V_n = (\mathrm{u}_n, \mathrm{u}_n', \mathrm{v}_n, \mathrm{v}_n', \mathrm{w}_n, \mathrm{w}_n', \mathrm{z}_n)^T$, from $V_n = \lambda_n U_n - \mathcal{A}U_n$ we get

$$\mathrm{u}_n = \lambda_n u_n - u_n', \tag{40}$$

$$\mathrm{u}_n' = \lambda_n u_n' - \frac{1}{\rho_1 h_1}\left[E_1 h_1 u_{n,xx} + k(-u_n + v_n + \alpha w_{n,x}) - \mu_1 u_n' - \mu_2 z_n(1)\right], \tag{41}$$

$$\mathrm{v}_n = \lambda_n v_n - v_n', \tag{42}$$

$$\mathrm{v}'_n = \lambda_n v_n' - \frac{1}{\rho_3 h_3}\left[E_3 h_3 v_{n,xx} - k(-u_n + v_n + \alpha w_{n,x}) - v_n'\right], \tag{43}$$

$$\mathrm{w}_n = \lambda_n w_n - w_n', \tag{44}$$

$$\mathrm{w}'_n = \lambda_n w_n' - \frac{1}{\rho h}\left[-EI w_{n,xxxx} + \alpha k(-u_n + v_n + \alpha w_{n,x})_x - w_n'\right], \tag{45}$$

$$\mathrm{z}_n = \lambda_n z_n + \frac{1}{\tau} z_{\rho,n}. \tag{46}$$

As $\mathrm{u}_n \to 0, \mathrm{v}_n \to 0, \mathrm{w}_n \to 0$, using (36) in (40), (37) in (42), (38) in (44) we obtain respectively $\lambda_n u_n \to 0, \lambda_n v_n \to 0, \lambda_n w_n \to 0$. Taking into account that $\lambda_n \to \infty$ we obtain that

$$u_n \to 0 \quad \text{as} \;\; n \to \infty, \tag{47}$$

$$v_n \to 0 \quad \text{as} \;\; n \to \infty, \tag{48}$$

$$w_n \to 0 \quad \text{as} \;\; n \to \infty. \tag{49}$$

Combining the convergences (36), (37), (38), (39), (47), (48), (49) with (41), (43), (45) we deduce that

$$E_1 h_1 u_{n,x} \to k\alpha w_n \quad \text{as} \;\; n \to \infty,$$

$$E_3 h_3 v_{n,x} \to -k\alpha w_n \quad \text{as} \;\; n \to \infty,$$

$$EI w_{n,xx} \to k\alpha w_n \quad \text{as} \;\; n \to \infty,$$

that leads to

$$u_{n,x} \to 0 \quad \text{as} \;\; n \to \infty,$$

$$v_{n,x} \to 0 \quad \text{as} \;\; n \to \infty,$$

$$w_{n,xx} \to 0 \quad \text{as} \;\; n \to \infty.$$

Poincaré inequality $||w_{n,x}|| \leq C||w_{n,xx}||$ implies

$$w_{n,x} \to 0 \quad \text{as} \;\; n \to \infty.$$

Multiplying (46) with z_n and integrating on $(0,1)$ we obtain

$$\lambda_n||z_n||^2_{L^2(0,1;L^2(0,L))} = \int_0^1 \langle \mathrm{z}_n \, , \, z_n \rangle \, \mathrm{d}\,\rho + \frac{1}{2\tau}||u_n'||^2 - \frac{1}{2\tau}||z_n(x,1,t)||^2. \tag{50}$$

Since z_n is bounded and z_n, $z_n(x,1,t)$, u_n' converge to zero, (50) leads to

$$||z_n||^2_{L^2(0,1;L^2(0,L))} \to 0.$$

Hence

$$\begin{aligned} ||U_n||^2_{\mathcal{H}} =& \rho_1 h_1||u_n'||^2 + E_1 h_1||u_{n,x}||^2 + \rho_3 h_3||v_n'||^2 + E_3 h_3||v_{n,x}||^2 + EI||w_{n,xx}||^2 \\ &+ \rho h||w_n'||^2 + k|| - u_n + v_n + \alpha w_{n,x}||^2 + \eta||z_n||^2_{L^2(0,1;L^2(0,L))} \to 0. \end{aligned}$$

We again have a contradiction and the proof of the theorem is complete.

□

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